WILSON LOOP APPROACH TO THE $q\bar{q}$ INTERACTION PROBLEM

N. BRAMBILLA and G. M. PROSPERI*
Dipartimento di Fisica dell'Università, Milano
INFN, Sezione di Milano, Via Celoria 16, 20133 Milano
E-mail: prosperi@mi.infn.it

ABSTRACT

It is shown that the semirelativistic $q\bar{q}$ potential, the relativistic flux tube model and a confining Bethe–Salpeter equation can be derived from QCD first principles in a unified point of view.

In this paper we want to show how, starting from the same standard evaluation of the Wilson loop integral and using similar techniques, it is possible to justify three different approaches to a treatment of the $q\bar{q}$ interaction on the basis of QCD alone, without making any ad hoc phenomenological hypothesis.

The three approaches are the derivation of a semirelativistic potential, the relativistic flux tube model and the Bethe–Salpeter equation. For simplicity, having here mainly a methodological purpose, we shall neglect spin in the last two cases. The modifications arising by the consideration of the quark spin shall be discussed in the paper presented by N. Brambilla [1] to which in a sense this one serves as an introduction.

As usual the Wilson loop integral is defined by

$$W = \frac{1}{3}\langle \mathrm{Tr}\mathrm{P}_\Gamma \exp ig\{\oint_\Gamma dx^\mu A_\mu\}\rangle \tag{1}$$

where the loop Γ is supposed made by a quark world line (Γ_1), an antiquark world line (Γ_2) and two straight lines connecting the initial and the final points of the two world lines (y_1, y_2 and x_1, x_2). The basic assumption is

$$i\ln W = i(\ln W)_{\mathrm{pert}} + \sigma S_{\mathrm{min}}\,, \tag{2}$$

$(\ln W)_{\mathrm{pert}}$ being the perturbative contribution to $\ln W$ and S_{min} the minimum area enclosed by Γ. At lowest order in g^2 one has

$$\begin{aligned} i(\ln W)_{\mathrm{pert}} \quad &= \frac{4}{3}g^2\int_{y_{10}}^{x_{10}} dt_1 \int_{y_{20}}^{x_{20}} dt_2 \frac{dz_1^\mu}{dt_1}\frac{dz_2^\nu}{dt_2} D_{\mu\nu}(z_1 - z_2) \\ &+\frac{2}{3}g^2\int_{y_{10}}^{x_{10}} dt_1 \int_{y_{10}}^{x_{10}} dt_1' \frac{dz_1}{dt_1}\frac{dz_1'}{dt_1'} D_{\mu\nu}(z_1 - z_1') + \end{aligned}$$

*Presenting author

$$
\begin{aligned}
&+\frac{2}{3}g^2\int_{y_{20}}^{x_{20}} dt_2 \int_{y_{20}}^{x_{20}} dt'_2 \frac{dz_2^\mu}{dt_2}\frac{dz_2'^\nu}{dt'_2} D_{\mu\nu}(z_2 - z'_2) = \\
&= \int_{t_i}^{t_f} \{ -\frac{4}{3}\frac{\alpha_s}{r}[1 - \frac{1}{2}(\delta^{hk} + \hat{r}^h\hat{r}^k)v_1^h v_2^k + \dots \}, \qquad (3)
\end{aligned}
$$

$D_{\mu\nu}$ being the gluon propagator; $z_1 = z_1(t_1)$ and $z_2 = z_2(t_2)$ the quark and the antiquark worldlines (with $z_{j0} = t_j$, $\mathbf{z}_j = \mathbf{z}_j(t_j)$, $z'_{j0} = t'_j$, $\mathbf{z}'_j = \mathbf{z}_j(t'_j)$ and $\mathbf{v}_1$ and $\mathbf{v}_2$ the corresponding velocities; $\mathbf{r} = \mathbf{z}_1 - \mathbf{z}_2$ the relative position). To make the second step in (3) one has to neglect the self–energy terms; to set $y_1^0 = y_2^0 = t_i$, $x_1^0 = x_2^0 = t_f$, $t_1 = t - \frac{\tau}{2}$, $t_2 = t - \frac{\tau}{2}$; to expand z_1 and z_2 in the relative time τ and to integrate over this. Furthermore, we use for $S_{\rm min}$ the straight line approximation, consisting in replacing $S_{\rm min}$ with the surface spanned by the straight lines connecting equal time points on the quark and the antiquark worldlines. In practice we write

$$
\begin{aligned}
S_{\rm min} \quad &\cong \int_{t_{\rm i}}^{t_{\rm f}} dt\, r \int_0^1 ds[1 - (s\frac{d\mathbf{z}_{1\rm T}}{dt} + (1-s)\frac{d\mathbf{z}_{2\rm T}}{dt})^2]^{\frac{1}{2}} = \\
&= \int_{t_i}^{t_f} dt[1 - \frac{1}{6}(\mathbf{v}_{1\rm T}^2 + \mathbf{v}_{2\rm T}^2 + \mathbf{v}_{1\rm T}\cdot\mathbf{v}_{2\rm T}) + \dots] \qquad (4)
\end{aligned}
$$

$\frac{d\mathbf{z}_{j\rm T}}{dt}$ and $\mathbf{v}_{j\rm T}$ denoting the transverse velocities : $v_{j\rm T}^h = (\delta^{hk} - \hat{r}^h\hat{r}^k)v_j^k$.

The basic object that we consider in our discussion is the usual gauge invariant quark–antiquark propagator

$$
\begin{aligned}
G_4^{gi}(x_1, x_2, y_1, y_2) &= \frac{1}{3}\langle 0|\mathrm{T}\psi_2^c(x_2)U(x_2,x_1)\psi_1(x_1)\overline{\psi}_1(y_1)U(y_1,y_2)\overline{\psi}_2^c(y_2)|0\rangle = \\
&= \frac{1}{3}\mathrm{Tr}\langle U(x_2,x_1)S_1(x_1,y_1;A)U(y_1,y_2)C^{-1}S_2(y_2,x_2;A)C\rangle \quad (5)
\end{aligned}
$$

where C denotes the charge-conjugation matrix, U the path-ordered gauge string

$$
U(b,a) = \mathrm{P}_{ba}\exp\left(ig\int_a^b dx^\mu\, A_\mu(x)\right) \qquad (6)
$$

(the integration path being the straight line joining a to b), S_1 and S_2 the single quark propagators in the external gauge field A^μ which are supposed to be defined by the equation

$$
[i\gamma^\mu(\partial_\mu - igA_\mu) - m]S(x,y;A) = \delta^4(x-y) \qquad (7)
$$

and the appropriate boundary conditions.

The various mentioned approaches to the treatment of the $q\bar{q}$ system correspond to different manipulations of (7) and (5). It is common to all cases the explicit resolution of (7) in terms of a path integral.

1. Semirelativistic potential [2,5]

For $x^0 > y^0$ by performing on (7) a Foldy–Wouthuysen transformation one can replace the 4×4 Dirac type propagator $S(x,y;A)$ by a 2×2 Pauli type propagator

$K(x,y;A)$ which satisfies a Schrödinger type equation with a hamiltonian expressed as a $\frac{1}{m^2}$ expansion. Solving this last equation by a path integration technique and using the expression so obtained in (5) (again for $x_1^0 = x_2^0 = t_f$, $y_1^0 = y_2^0 = t_i$ with $t_f - t_i > 0$ and large) one arrives eventually to the two–particle Pauli type propagator

$$
\begin{aligned}
K(\mathbf{x}_1,\mathbf{x}_2,\mathbf{y}_1,\mathbf{y}_2;t_\mathrm{f}-t_\mathrm{i}) &= \int_{\mathbf{z}_1(t_\mathrm{i})=\mathbf{y}_1}^{\mathbf{z}_1(t_\mathrm{f})=\mathbf{x}_1} \mathcal{D}\mathbf{z}_1\mathcal{D}\mathbf{p}_1 \int_{\mathbf{z}_2(t_\mathrm{i})=\mathbf{y}_2}^{\mathbf{z}_2(t_\mathrm{f})=\mathbf{x}_2} \mathcal{D}\mathbf{z}_2\mathcal{D}\mathbf{p}_2 \\
&\exp\{i\int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \sum_{j=1}^{2}[\mathbf{p}_j\cdot\dot{\mathbf{z}}_j - m_j - \frac{\mathbf{p}_j^2}{2m_j} + \frac{\mathbf{p}_j^4}{8m_j^3}]\}\langle\frac{1}{3}\mathrm{Tr}\,\mathrm{T_s}\,\mathrm{P}\exp\{ig\oint_\Gamma dx^\mu\, A_\mu(x) \\
&+\sum_{j=1}^{2}\frac{ig}{m_j}\int_{\Gamma_j} dx^\mu(S_j^l\hat{F}_{l\mu}(x) - \frac{1}{2m_j}S_j^l\varepsilon^{lkr}p_j^kF_{\mu r}(x) - \frac{1}{8m_j}D^\nu F_{\nu\mu}(x))\}\rangle\,. \qquad (8)
\end{aligned}
$$

where $\mathrm{T_s}$ is the time-ordering prescription for the spin matrices alone. The semirelativistic potential is obtained by comparing (8) with the path integral solution of the two particle Schrödinger equation, having used the second step of (3) and (4) and having reduced the spin dependent terms to functional derivatives of $\ln W$. The final result is

$$
\begin{aligned}
V &= -\frac{4}{3}\frac{\alpha_s}{r} + \sigma r - \frac{4}{3}\frac{\alpha_\mathrm{s}^2}{4\pi}\frac{1}{r}[\frac{66-4N_f}{3}(\ln\mu r + \gamma) + A] + \\
&+ \frac{1}{m_1m_2}\{\frac{4}{3}\frac{\alpha_s}{r}(\delta^{hk} + \hat{r}^h\hat{r}^k)p_1^hp_2^k\}_{\mathrm{Weyl\,ord}} \qquad (9) \\
&- \sum_{j=1}^{2}\frac{1}{6m_j^2}\{\sigma r\mathbf{p}_{j\mathrm{T}}\}_{\mathrm{Weyl\,ord}} - \frac{1}{6m_1m_2}\{\sigma r\mathbf{p}_{1\mathrm{T}}\cdot\mathbf{p}_{2\mathrm{T}}\}_{\mathrm{Weyl\,ord}} + \\
&+\frac{1}{8}\sum_{j=1,2}\frac{1}{m_j^2}\nabla^2\left(-\frac{4}{3}\frac{\alpha_s}{r} + \sigma r\right) + \left(\frac{4}{6}\frac{\alpha_s}{r^3} - \frac{\sigma}{2r}\right)\sum_{j=1,2}\frac{1}{m_j^2}\mathbf{S}_j\cdot\mathbf{L}_j + \frac{1}{m_1m_2}\frac{4}{3}\frac{\alpha_s}{r^3}\times \\
&(\mathbf{S}_2\cdot\mathbf{L}_1 + \mathbf{S}_1\cdot\mathbf{L}_2) + \frac{4\alpha_\mathrm{s}}{3m_1m_2}\left[(\frac{3}{r^5}(\mathbf{S}_1\cdot\mathbf{r})(\mathbf{S}_2\cdot\mathbf{r}) - \frac{\mathbf{S}_1\cdot\mathbf{S}_2}{r^3}) + \frac{8\pi}{3}\delta^3(\mathbf{r})\mathbf{S}_1\cdot\mathbf{S}_2\right]
\end{aligned}
$$

2. Relativistic flux tube model [6]

Let us neglect in Eq.(8) the spin–dependent terms and replace the $\frac{1}{m^2}$ expansion of the kinetic term by its exact relativistic expression

$$
\begin{aligned}
&K(\mathbf{x}_1,\mathbf{x}_2;\mathbf{y}_1,\mathbf{y}_2;t_\mathrm{f}-t_\mathrm{i}) = \\
&\int\mathcal{D}\mathbf{z}_1\mathcal{D}\mathbf{p}_1\int\mathcal{D}\mathbf{z}_2\mathcal{D}\mathbf{p}_2\exp\left\{i\left[\int_{t_\mathrm{i}}^{t_\mathrm{f}}dt\sum_{j=1}^{2}(\mathbf{p}_j\cdot\dot{\mathbf{z}}_j - \sqrt{m_j^2+\mathbf{p}_j^2})\right] + \ln W\right\}. \qquad (10)
\end{aligned}
$$

Let us also neglect for simplicity $i(\ln W)_\mathrm{pert}$ in (2) and assume that a sensible approximation is obtained even in the relativistic case postulating the first line of (4)

in the center–of–mass frame of the two particles. Then, integrating on the momenta one obtains the ordinary lagrangian

$$L = -\sum_{j=1}^{2} m_j \sqrt{1-\dot{\mathbf{z}}_j^2} - \sigma r \int_0^1 ds [1-(s\dot{\mathbf{z}}_{1\mathrm{T}} + (1-s)\dot{\mathbf{z}}_{2\mathrm{T}})^2]^{1/2}. \tag{11}$$

This coincides with the relativistic flux–tube model lagrangian [6].

From (12) is not possible to obtain even a classical hamiltonian in a closed form, due to the complicate velocity dependence. However, in terms of an expansion in $\frac{\sigma}{m^2}$ we have (we assume $m_1 = m_2 = m$ for simplicity)

$$\mathcal{H}(\mathbf{r},\mathbf{q}) = 2\sqrt{m^2+q^2} + \frac{\sigma r}{2}\Big[\frac{\sqrt{m^2+q^2}}{q_{\mathrm{T}}} \arcsin\frac{q_{\mathrm{T}}}{\sqrt{m^2+q^2}} + \sqrt{\frac{m^2+q_{\mathrm{r}}^2}{m^2+q^2}}\Big] + \ldots \tag{12}$$

with $\mathbf{r} = \mathbf{z}_{1\mathrm{CM}} - \mathbf{z}_{2\mathrm{CM}}$, $\mathbf{q} = \mathbf{p}_{1\mathrm{CM}} = -\mathbf{p}_{2\mathrm{CM}}$, $\mathbf{q}_{\mathrm{r}} = (\hat{\mathbf{r}}\cdot\mathbf{q})/\hat{\mathbf{r}}$ and $q_{\mathrm{T}}^h = (\delta^{hk} - \hat{r}^h\hat{r}^k)q^k$. From this a quantum hamiltonian can be immediately obtained by setting

$$\langle \mathbf{k}'|H_{\mathrm{FT}}|\mathbf{k}\rangle = \int \frac{d\mathbf{r}}{(2\pi)^3} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}\, \mathcal{H}(\mathbf{r}, \frac{\mathbf{k}'+\mathbf{k}}{2}), \tag{13}$$

Then by an expansion in $\frac{1}{m^2}$ a semirelativistic hamiltonian can be obviously recovered with a confining potential given by the spin–independent part of (9).

3. Bethe–Salpeter equation [7,8]

Let us go back to the quantity analogous to (5) for spinless quarks and in it use the covariant representation for the quark propagator in an external gauge field

$$\Delta(x,y|A) = \frac{-i}{2}\int_0^\infty d\tau \int_{z(0)=y}^{z(\tau)=x} \mathcal{D}z \mathrm{Pexp}\, i\int_0^\tau d\tau'\{-\frac{1}{2}[(\frac{dz}{d\tau'})^2 + m^2] - g z^{\mu\prime} A_\mu(z)\} \tag{14}$$

In place of (8) we find

$$\begin{aligned} G_4(x_1,x_2;y_1,y_2) = (\frac{-i}{2})^2 \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_{z_1(0)=y}^{z_1(t_1)=x_1} \mathcal{D}z_1 \int_{z_2(0)=y_2}^{z_2(\tau_2)=x_2} \mathcal{D}z_2 \\ \exp\{\frac{-i}{2}\int_0^{\tau_1} d\tau_1'[(\frac{dz_1}{d\tau_1'})^2 + m_1^2] - \frac{i}{2}\int_0^{\tau_2} d\tau_2'[(\frac{dz_2}{d\tau_2'})^2 + m_2^2] + \ln W\} \end{aligned} \tag{15}$$

where the path connecting y with x is now written as $z^\mu = z^\mu(\tau)$, in terms of an independent parameter τ (rather than the time t) and z' stands for $z(\tau')$. Then assuming again the first steps of (3) and (4) in the center of mass frame (after rewriting in terms of τ_1 and τ_2), replacing them in (5) and performing appropriate manipulations, one can obtain an inhomogeneous Bethe–Salpeter equation with a kernel resulting by

the sum $I = I_{\rm pert} + I_{\rm conf}$ of a perturbative part and a confinement one. This last in the momentum representation can be written as

$$\hat{I}_{\rm conf}(p_1', p_2'; p_1, p_2) = \frac{1}{(2\pi)^3} \int d^3\mathbf{r} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} J(\mathbf{r}, \frac{p_1' + p_1}{2}, \frac{p_2' + p_2}{2}) \tag{16}$$

($p_1' + p_2' = p_1 + p_2$, $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{k}$, $\mathbf{p}_1' = -\mathbf{p}_2' = \mathbf{k}'$) with

$$\begin{aligned} J(\mathbf{r}, q_1, q_2) &= (2\pi)^3 \frac{\sigma r}{2} \frac{1}{q_{10} + q_{20}} [q_{20}^2 \sqrt{q_{10}^2 - \mathbf{q}_{\rm T}^2} + q_{10}^2 \sqrt{q_{20}^2 - \mathbf{q}_{\rm T}^2} + \\ &+ \frac{q_{10}^2 q_{20}^2}{|\mathbf{q}_{\rm T}|} (\arcsin \frac{|\mathbf{q}_{\rm T}|}{|q_{10}|} + \arcsin \frac{|\mathbf{q}_{\rm T}|}{|q_{20}|})] + O(\frac{\sigma^2}{m^4}) \end{aligned} \tag{17}$$

Essential steps in the derivation are the equation

$$S_{\rm min} = \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 \delta(z_{10} - z_{20}) |\mathbf{z}_1 - \mathbf{z}_2| \int_0^1 ds \{\dot{z}_{10}^2 \dot{z}_{20}^2 - (s \mathbf{z}_{1\rm T} \dot{z}_{20} + (1-s) \dot{\mathbf{z}}_{2\rm T} \dot{z}_{10})^2\}^{\frac{1}{2}} \tag{18}$$

equivalent to (3) and the recurrence identity

$$\exp i \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 f(z_1, z_2) = 1 + i \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 f(z_1, z_2) \exp i \int_0^{\tau_1} d\tau_1' \int_0^{s_2} d\tau_2' f(z_1', z_2'). \tag{19}$$

Notice that, according to a standard procedure, the BS kernel $\hat{I}$ can be associated with a relativistic potential (to be used in the Salpeter equation) given by

$$\langle \mathbf{k}' | V | \mathbf{k} \rangle = \frac{1}{(2\pi)^3} \frac{m_1 m_2}{\sqrt{w_1(\mathbf{k}) w_2(\mathbf{k}) w_1(\mathbf{k}') w_2(\mathbf{k}')}} \hat{I}_{\rm inst}(\mathbf{k}', \mathbf{k}) \tag{20}$$

where $w_j(\mathbf{k}) = \sqrt{m_j^2 + \mathbf{k}^2}$ and the instantaneous kernel $\hat{I}_{\rm inst}$ is obtained from $\hat{I}$ by setting $p_{j0} = p_{jo}' = \frac{1}{2}(w_j(\mathbf{k}) + w_j(\mathbf{k}')$. Obviously the resulting hamiltonian coincides with (12), (13).

4. References

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